Int J Theor Phys (2009) 48: 1545–1553
DOI 10.1007/s10773-009-9926-z

# Entangled State in Quantization of Magnetic Flux Qubits with Mutual Inductance Coupling

**Bao-Long Liang · Ji-Suo Wang · Xiang-Guo Meng · Jie Su**

Received: 27 October 2008 / Accepted: 6 January 2009 / Published online: 14 January 2009


**Abstract** Via the Hamilton dynamical approach we have constructed Hamiltonian for the mutual inductance coupling magnetic flux qubits. The entangled state representation is used to propose Cooper-pair number-phase quantization and the Hamiltonian operator for the whole system. The dynamical evolution of the phase difference operator and the Cooper-pairs number operator is investigated by virtue of Heisenberg equations.

**Keywords** Josephson junction · Flux qubit · Phase operator

## 1 Introduction

In recent years, with the progress on quantum computation and quantum information, the mesoscopic circuit including Josephson junctions has been paid much attention [1–10] because of the magic quantum tunneling effect, which makes them potential qubits: charge qubits, phase qubits and flux qubits. In flux qubits the flux is the appropriate quantum degree of freedom. The super-conducting loop including a Josephson junction can be referred to as the simplest flux qubits, which can be operated by a magnetic flux through the loop [11]. Certainly, first of all investigations on the mesoscopic scale circuit is the quantization of the circuit. Vourdas took the quantization of such a flux qubit structure by considering the charge as the generalized momentum and the phase as the generalized coordinate [12, 13], as is reckoned as canonical quantization method.

In this paper, we will couple two single-flux-qubit structures by mutual-inductance, then quantize the system using the entangled state representation [14–18]. The Josephson junction is comprised of two superconductors "weakly" connected by a thin layer of insulating material. Feynman analyzed such a junction by assuming [19] "electron pairs are

Project 10574060 supported by the National Natural Science Foundation of China and project X071045 supported by the Science Foundation of Liaocheng University.

B.-L. Liang (✉) · J.-S. Wang · X.-G. Meng · J. Su
School of Physics Science and Information Engineering, Liaocheng University, Shandong 252059, People's Republic of China
e-mail: baolong-liang951@sohu.com

bosons, ..., nearly all of the pairs will be locked down at the lowest energy in exactly the same state". Then he assigned the super-conductivity state in each plate the wave function $\psi_l = \sqrt{\rho_l} e^{i\phi_l}$, where $\phi_l$ $(l = 1, 2)$ is the phase on the $l$th plate of junction and $\rho_l$ is the density of electrons. Based on the above assumption, we will propose Cooper-pair number-phase quantization scheme by the entangled state representation. Our procedures are as follows: Firstly, we analyse the circuit from the point of view of classical Hamilton dynamics and obtain the classical Hamiltonian. Secondly, we give the Cooper-pair number-phase quantization scheme for this circuit. Finally, we investigate the time evolution of the phase difference operator and the Cooper-pairs number operator.

## 2 Hamiltonian for the Mutual-Inductance Coupling Magnetic Flux Qubits

Considering the mesoscopic circuit (see Fig. 1), it consists of two uniform super-conducting loops (each with self-inductance $L_l$ $(l = 1, 2)$), which are coupled by the mutual-inductance $M$ between them. Each loop includes a Josephson junction with the capacitance $C_l$ $(l = 1, 2)$ and the coupling energy $E^0_{jl}$ $(l = 1, 2)$. In fact, we can refer to the mesoscopic circuit as two magnetic flux qubits coupled by mutual-inductance. In the following, we will analyse this circuit from the point of view of classical Hamilton dynamics.

A single Josephson junction is characterized by two Josephson equations (Josephson current equation and voltage equation)

$$I_j = I_c \sin\varphi_j, \quad \dot{\varphi}_j = \frac{2eu_j}{\hbar}, \tag{1}$$

where the subscript $j$ represents the Josephson junction, $I_c = 2eE^0_j/\hbar$ is the critical electric current of the junction, $u_j$ is the voltage drop across the Josephson junction and $\varphi_j$ is the phase difference across the two plates of a junction. Using Ginzburg-Landau equation the phase difference, which is generated by the magnetic flux $\Phi_l$ through the $l$th superconducting loop, is

$$\varphi_{l(H)} = \frac{2e}{\hbar}\oint \vec{\mathcal{A}}_l \cdot d\vec{s}_l = \frac{2e}{\hbar}\Phi_l \quad (l = 1, 2), \tag{2}$$

where $\vec{\mathcal{A}}_l$ is the vector potential induced by the magnetic flux $\Phi_l$. So $\varphi_{l(H)}$ and $\varphi_l$, which represents the phase difference across the two plates of the $l$th junction, satisfies the following relation

$$\varphi_l - \varphi_{l(H)} = 2\pi n \quad (n \text{ being an integer}) \tag{3}$$

which leads to

$$\varphi_l = 2\pi n + \frac{2e}{\hbar}\Phi_l. \tag{4}$$

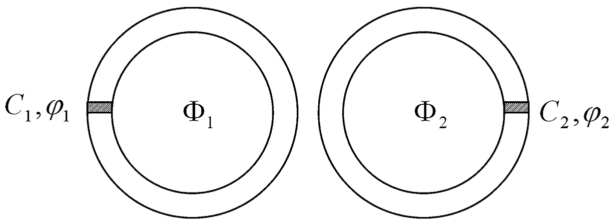


**Fig. 1** The magnetic flux qubits with mutual-inductance coupling

And $\Phi_1$ and $\Phi_2$, respectively, satisfy the following relations

$$\Phi_1 = \Phi_{x1} + I_1 L_1 + I_2 M, \qquad \Phi_2 = \Phi_{x2} + I_2 L_2 + I_1 M, \tag{5}$$

where $\Phi_{xl}$ ($l = 1, 2$) is the external classical magnetic flux through the $l$th super-conducting loop and $M = k\sqrt{L_1 L_2}$, and $k$ represents the coupling constant between the two loops. From (5) we obtain

$$I_1 = \frac{(\Phi_1 - \Phi_{x1}) - M(\Phi_2 - \Phi_{x2})/L_2}{L_1(1-k^2)}, \qquad I_2 = \frac{(\Phi_2 - \Phi_{x2}) - M(\Phi_1 - \Phi_{x1})/L_1}{L_2(1-k^2)}. \tag{6}$$

If considering the phase difference $\varphi_l$ as the generalized coordinate, from (4)–(6), we can obtain the potential energy, related to $\varphi_l$ (or $\Phi_l$), of the system

$$\begin{aligned} \mathcal{V} &= \frac{1}{2}L_1 I_1^2 + \frac{1}{2}L_2 I_2^2 + M I_1 I_2 + \sum_{l=1}^{2} E_{jl}^0 (1 - \cos\varphi_l) \\ &= \frac{1}{2}\sum_{(l\neq m)=1}^{2} L_l \left[\frac{(\Phi_l - \Phi_{xl}) - \frac{M}{L_m}(\Phi_m - \Phi_{xm})}{L_l(1-k^2)}\right]^2 \\ &\quad + M \prod_{(l\neq m)=1}^{2} \frac{(\Phi_l - \Phi_{xl}) - \frac{M}{L_m}(\Phi_m - \Phi_{xm})}{L_l(1-k^2)} + \sum_{l=1}^{2} E_{jl}^0 (1 - \cos\varphi_l). \end{aligned} \tag{7}$$

On the other hand, the charging energy stored in all the capacitors is

$$\mathcal{T} = \frac{1}{2}C_1 u_{j1}^2 + \frac{1}{2}C_2 u_{j2}^2, \tag{8}$$

where $\mathcal{T}$ is referred to as the kinetic energy, since from (1) one can see $u_{j1}$ and $u_{j2}$ are, respectively, related to $\varphi_1$ and $\varphi_2$. Substituting (1) into (8) leads to

$$\mathcal{T} = \frac{1}{2}C_1\left(\frac{\hbar}{2e}\right)^2 \dot{\varphi}_1^2 + \frac{1}{2}C_2\left(\frac{\hbar}{2e}\right)^2 \dot{\varphi}_2^2. \tag{9}$$

Then the Lagrangian of the system is

$$\mathcal{L} = \mathcal{T} - \mathcal{V}. \tag{10}$$

Thus, we get the generalized momentum, which is conjugate to $\varphi_l$

$$p_l = \frac{\partial \mathcal{L}}{\partial \dot{\varphi}_l} = \left(\frac{\hbar}{2e}\right)^2 C_l \dot{\varphi}_l = \frac{\hbar}{2e} q_l = \mathcal{N}_l \hbar, \tag{11}$$

where we have supposed $q_l = 2\mathcal{N}_l e$, noting that $p_l$ is proportional to the Cooper-pairs number $\mathcal{N}_l$ on the $l$th junction, which implies the possibility of number-phase quantization, as we will carry out in the following. Combining (10)–(11) we obtain the Hamiltonian of the system

$$\mathcal{H} = \sum_{i=1}^{2} p_l \dot{\varphi}_l - \mathcal{L}$$

$$= \sum_{l=1}^{2}\left[E_{cl}\mathcal{N}_l^2 + \frac{(\Phi_l - \Phi_{xl})^2}{2L_l(1-k^2)} + E_{jl}^0(1-\cos\varphi_l)\right] - \prod_{l=1}^{2}\frac{k(\Phi_l - \Phi_{xl})}{\sqrt{L_l}(1-k^2)}, \tag{12}$$

where $E_{cl} = \frac{(2e)^2}{2C_l}$ is the single Cooper-pair charging energy on the $l$th junction.

## 3 Cooper-Pairs Number-Phase Quantization for the System

Now we will quantize the Hamiltonian given by (12). For this purpose, following Feynman's idea that "electron pairs are bosons, ..., nearly all of the pairs will be locked down at the lowest energy in exactly the same state", "a bound pair acts as a Bose particle", we are naturally led to provide the Bose operator model. As is mentioned above, the Cooper-pairs number $\mathcal{N}_l$ should be quantized, so we introduce the two-mode Bose operator

$$\hat{\mathcal{N}}_l \equiv \hat{a}_l^\dagger \hat{a}_l - \hat{b}_l^\dagger \hat{b}_l, \tag{13}$$

where $\hat{a}_l^\dagger \hat{a}_l - \hat{b}_l^\dagger \hat{b}_l$ is the number-difference between the super-conductors of the $l$th Josephson junction. This form is like the charge operator in the complex-scalar quantized field theory [20], in which the quantized scalar field operators are expanded as

$$\begin{aligned}\hat{\phi}(x) &= \int d^3p[\hat{a}_p u_p(\vec{x},t) + \hat{b}_p^\dagger u_p^*(\vec{x},t)],\\ \hat{\phi}^\dagger(x) &= \int d^3p[\hat{a}_p^\dagger u_p^*(\vec{x},t) + \hat{b}_p u_p(\vec{x},t)],\end{aligned} \tag{14}$$

where $\hat{a}_p$ denotes positive charge annihilation and $\hat{b}_p^\dagger$ denotes negative charge creation, and

$$u_p(\vec{x},t) = \frac{1}{\sqrt{2\omega_p(2\pi)^3}}\exp[-i(\omega_p t - px)], \quad \omega_p = \sqrt{p^2+m^2}, \tag{15}$$

where we have supposed $\hbar = c = 1$. The charge operator of the field is

$$\int d^3p(\hat{a}_p^\dagger \hat{a}_p - \hat{b}_p^\dagger \hat{b}_p). \tag{16}$$

To perform the number-phase quantization we introduce the entangled state $|\eta_l\rangle$ [14–18]

$$|\eta_l\rangle = \exp\left[-\frac{1}{2}|\eta_l|^2 + \eta_l a_l^\dagger - \eta_l^* b_l^\dagger + a_l^\dagger b_l^\dagger\right]|00\rangle_l\,, \tag{17}$$

where $\eta_l = |\eta_l|e^{i\varphi_l}$, and $|00\rangle_l$ is the two-mode vacuum state. The $|\eta_l\rangle$ state is constructed based on the idea of quantum entanglement of Einstein, Podolsky and Rosen [21]. Using $[\hat{a}_l, \hat{a}_l^\dagger] = [\hat{b}_l, \hat{b}_l^\dagger] = 1$, we see that $|\eta_l\rangle$ obeys the eigenvector equations

$$(\hat{a}_l - \hat{b}_l^\dagger)|\eta_l\rangle = \eta_l|\eta_l\rangle\,,\ (\hat{b}_l - \hat{a}_l^\dagger)|\eta_l\rangle = -\eta_l^*|\eta_l\rangle\,. \tag{18}$$

The set of $|\eta_l\rangle$ makes up a complete representation

$$\int \frac{d^2\eta_l}{\pi}|\eta_l\rangle\langle\eta_l| = 1. \tag{19}$$

We can construct the bosonic phase operator for Josephson junction [6, 22, 23]

$$e^{i\hat{\varphi}_l}=\sqrt{\frac{\hat{a}_l-\hat{b}_l^\dagger}{\hat{a}_l^\dagger-\hat{b}_l}},\qquad e^{-i\hat{\varphi}_l}=\sqrt{\frac{\hat{a}_l^\dagger-\hat{b}_l}{\hat{a}_l-\hat{b}_l^\dagger}},\qquad \cos\hat{\varphi}_l=\frac{1}{2}(e^{i\hat{\varphi}_l}+e^{-i\hat{\varphi}_l}),\tag{20}$$

because in the $|\eta_l\rangle$ representation, $e^{i\hat{\varphi}_l}$ behaves a phase

$$e^{i\hat{\varphi}_l}|\eta_l\rangle=e^{i\varphi_l}|\eta_l\rangle,\qquad e^{-i\hat{\varphi}_l}|\eta_l\rangle=e^{-i\varphi_l}|\eta_l\rangle.\tag{21}$$

Note that $[\hat{a}_l^\dagger-\hat{b}_l,\hat{a}_l-\hat{b}_l^\dagger]=0$, so they can reside in the same square root. It then follows $\hat{\varphi}_l=\frac{1}{2i}\ln\frac{\hat{a}_l-\hat{b}_l^\dagger}{\hat{a}_l^\dagger-\hat{b}_l}$, $\hat{\varphi}_l|\eta_l\rangle=\varphi_l|\eta_l\rangle$. From (18) we can derive

$$\begin{aligned}\hat{\mathcal{N}}_l|\eta_l\rangle&\equiv(\hat{a}_l^\dagger\hat{a}_l-\hat{b}_l^\dagger\hat{b}_l)|\eta_l\rangle\\&=[\hat{a}_l^\dagger(\eta_l+\hat{b}_l^\dagger)-\hat{b}_l^\dagger(\hat{a}_l^\dagger-\eta_l^*)]|\eta_l\rangle\\&=|\eta_l|(\hat{a}_l^\dagger e^{i\varphi_l}+\hat{b}_l^\dagger e^{-i\varphi_l})|\eta_l\rangle\\&=-i\frac{\partial}{\partial\varphi_l}|\eta_l\rangle.\end{aligned}\tag{22}$$

Thus

$$\begin{aligned}&\left[\hat{\varphi}_l,\hat{\mathcal{N}}_l\right]|\eta_l\rangle=[\varphi_l,-i\frac{\partial}{\partial\varphi_l}]|\eta_l\rangle=i|\eta_l\rangle\rightarrow\left[\hat{\varphi}_l,\hat{\mathcal{N}}_l\right]=i,\\&\left[\hat{\mathcal{N}}_l,\cos\hat{\varphi}_l\right]=i\sin\hat{\varphi}_l,\qquad\left[\hat{\mathcal{N}}_l,\sin\hat{\varphi}_l\right]=-i\cos\hat{\varphi}_l,\end{aligned}\tag{23}$$

which embodies the number-phase quantization. When making analogy with the classical case given by (3)–(4), we can introduce the following bosonic magnetic flux operator

$$\hat{\Phi}_l=\frac{\hbar}{2e}(\hat{\varphi}_l-2\pi n)=\frac{\hbar}{2e}\left(\frac{1}{2i}\ln\frac{\hat{a}_l-\hat{b}_l^\dagger}{\hat{a}_l^\dagger-\hat{b}_l}-2\pi n\right).\tag{24}$$

Based on (23)–(24), the following commutative relation can be deduced

$$[\hat{\Phi}_l,\hat{\mathcal{N}}_l]=\frac{i\hbar}{2e}.\tag{25}$$

As a consequence of (23)–(25), the classical Hamiltonian given by (12) is quantized as

$$\hat{\mathcal{H}}=\sum_{l=1}^{2}\left[E_{cl}\hat{\mathcal{N}}_l^2+\frac{(\hat{\Phi}_l-\Phi_{xl})^2}{2L_l(1-k^2)}+E_{jl}^0\left(1-\cos\hat{\varphi}_l\right)\right]-\prod_{l=1}^{2}\frac{k(\hat{\Phi}_l-\Phi_{xl})}{\sqrt{L_l}(1-k^2)}.\tag{26}$$

## 4 Time Evolution of the Phase Difference Operator and the Cooper-Pairs Number Operator

In this section we shall discuss the time evolution of phase difference operator and the Cooper-pairs number operator. For the 1st Josephson junction, by virtue of Heisenberg equations of motion and (24)–(26) we deduce

$$\hat{I}_1' = -\frac{2ed\hat{\mathcal{N}}_1}{dt} = \frac{-2e}{i\hbar}\left[\hat{\mathcal{N}}_1, \hat{\mathcal{H}}\right]$$
$$= I_{c1}\sin\hat{\varphi}_1 + \frac{\hat{\Phi}_1 - \Phi_{x1}}{L_1(1-k^2)} - \frac{k(\hat{\Phi}_2 - \Phi_{x2})}{\sqrt{L_1L_2}(1-k^2)}, \tag{27}$$

$$\frac{d\hat{\varphi}_1}{dt} = \frac{1}{i\hbar}\left[\hat{\varphi}_1, \hat{\mathcal{H}}\right] = \frac{2E_{cl}}{\hbar}\hat{\mathcal{N}}_1 = \frac{2e}{\hbar}\hat{u}_1, \tag{28}$$

where $\hat{u}_1 = \frac{2e}{C_l}\hat{\mathcal{N}}_1 = \frac{2e}{C_l}(\hat{a}_1^\dagger\hat{a}_1 - \hat{b}_1^\dagger\hat{b}_1)$, which can be referred to as Josephson voltage operator equation in bosonic form. Similarly, for the 2nd Josephson junction, we have

$$\hat{I}_2' = -\frac{2ed\hat{\mathcal{N}}_1}{dt} = I_{c2}\sin\hat{\varphi}_2 + \frac{\hat{\Phi}_2 - \Phi_{x2}}{L_2(1-k^2)} - \frac{k(\hat{\Phi}_1 - \Phi_{x1})}{\sqrt{L_1L_2}(1-k^2)}, \tag{29}$$

$$\frac{d\hat{\varphi}_2}{dt} = \frac{2E_{c2}}{\hbar}\hat{\mathcal{N}}_2 = \frac{2e}{\hbar}\hat{u}_2, \tag{30}$$

where $\hat{u}_2 = \frac{2e}{C_l}\hat{\mathcal{N}}_2 = \frac{2e}{C_l}(\hat{a}_2^\dagger\hat{a}_2 - \hat{b}_2^\dagger\hat{b}_2)$. From (27) and (29) we can see that with the time evolving the Cooper-pairs move from one polar plate of the Josephson junction to another along two main paths: the super-conducting tunnel (exhibiting tunnelling current) and the super-conducting loop (exhibiting self-inductance current and mutual-inductance current). When the Josephson junction is radiated by extra external energy, the inductive charge will accumulate on the plates and affect the phase difference.

## 5 Phase Difference Time Evolution when Josephson Junction is Radiated by Extra Energy

In the following we reveal the phase difference time evolution when extra energy is applied to the 1st Josephson junction. For this aim, we consider the work performed by $\hat{I}_1'$ in a short time interval $\Delta t$,

$$\hat{u}_1\hat{I}_1'\Delta t = \frac{2e}{C_1}\hat{\mathcal{N}}_1\left[I_{c1}\sin\hat{\varphi}_1 + \frac{\hat{\Phi}_1 - \Phi_{x1}}{L_1(1-k^2)} - \frac{k(\hat{\Phi}_2 - \Phi_{x2})}{\sqrt{L_1L_2}(1-k^2)}\right]\Delta t, \tag{31}$$

which implies that when extra energy (say, light radiation) is applied to the junction, the corresponding Hamilton operator can be expressed in the following form

$$\mathcal{H}' = \lambda_1\hat{\mathcal{N}}_1\left[I_{c1}\sin\hat{\varphi}_1 + \frac{\hat{\Phi}_1 - \Phi_{x1}}{L_1(1-k^2)} - \frac{k(\hat{\Phi}_2 - \Phi_{x2})}{\sqrt{L_1L_2}(1-k^2)}\right], \tag{32}$$

where $\lambda_1$ is a coupling constant, and the constant $\frac{2e}{C_1}$ is absorbed into $\lambda_1$. Taking $\mathcal{H}'$ in the interaction picture and according to the equation of motion in this picture, we deduce

$$\frac{d\hat{\Phi}_2}{dt} = \frac{1}{i\hbar}[\hat{\Phi}_2, \mathcal{H}'] = 0, \tag{33}$$

$$\frac{d\sin\hat{\varphi}_1}{dt} = \frac{1}{i\hbar}\left[\sin\hat{\varphi}_1, \hat{\mathcal{H}}'\right]$$
$$= \frac{\lambda_1\cos\hat{\varphi}_1}{\hbar}\left[I_{c1}\sin\hat{\varphi}_1 + \frac{\hat{\Phi}_1 - \Phi_{x1}}{L_1(1-k^2)} - \frac{k(\hat{\Phi}_2 - \Phi_{x2})}{\sqrt{L_1L_2}(1-k^2)}\right], \quad (34)$$

$$\frac{d\cos\hat{\varphi}_1}{dt} = \frac{1}{i\hbar}\left[\cos\hat{\varphi}_1, \hat{\mathcal{H}}'\right]$$
$$= -\frac{\lambda_1\sin\hat{\varphi}_1}{\hbar}\left[I_{c1}\sin\hat{\varphi}_1 + \frac{\hat{\Phi}_1 - \Phi_{x1}}{L_1(1-k^2)} - \frac{k(\hat{\Phi}_2 - \Phi_{x2})}{\sqrt{L_1L_2}(1-k^2)}\right], \quad (35)$$

then it follows

$$\frac{d}{dt}\tan\frac{\hat{\varphi}_1}{2} = \frac{d}{dt}\left(\frac{1-\cos\hat{\varphi}_1}{\sin\hat{\varphi}_1}\right)$$
$$= \mu_1\tan\frac{\hat{\varphi}_1}{2} + \left[\frac{\mu_2}{2}(\hat{\Phi}_1 - \Phi_{x1}) + \frac{\mu_3}{2}(\hat{\Phi}_2 - \Phi_{x2})\right]\left(1 + \tan^2\frac{\hat{\varphi}_1}{2}\right), \quad (36)$$

where

$$\mu_1 = \frac{\lambda_1 I_{c1}}{\hbar}, \qquad \mu_2 = \frac{\lambda_1}{L_1(1-k^2)\hbar}, \qquad \mu_3 = -\frac{\lambda_1 k}{\sqrt{L_1L_2}(1-k^2)\hbar}. \quad (37)$$

From (36) we can see how the magnetic flux through the super-conducting loop affects the time evolution of the phase difference after external radiation, as is helpful for us to analyse the quantum manipulation, since we may think it is actually to manipulate the phase difference of Josephson junction to control the qubit by magnetic flux. Such effect can be obvious from the general solution of (36). In order to obtain the general solution, let

$$y \equiv \tan\frac{\hat{\varphi}_1}{2}, \qquad r(t) = p(t) \equiv \frac{\mu_2}{2}(\hat{\Phi}_1 - \Phi_{x1}) + \frac{\mu_3}{2}(\hat{\Phi}_2 - \Phi_{x2}), \qquad q(t) \equiv \mu_1. \quad (38)$$

Substituting (38) into (36) leads to

$$\frac{dy}{dt} = p(t)y^2 + q(t)y + r(t), \quad (39)$$

which is the standard Riccati equation, where $p(t) \neq 0$, $r(t) \neq 0$. Here, we suppose $y_1(0) \equiv \tan\frac{\hat{\varphi}_1(0)}{2}$ is a particular solution, and make a transformation $y = y_1(0) + \frac{1}{u}$, then (39) turns into the following form

$$\frac{du}{dt} + [q(t) + 2p(t)y_1(0)]u = -p(t), \quad (40)$$

which is linear differential equation, and its general solution is

$$u = \exp(-\chi)\left[\int\left(\beta - \frac{1}{2}\mu_2\hat{\Phi}_1\right)\exp(\chi)dt + c\right], \quad (41)$$

where

$$\beta = \frac{1}{2}\mu_2\Phi_{x1} - \frac{1}{2}\mu_3(\hat{\Phi}_2 - \Phi_{x2}), \qquad \alpha = \mu_1 - 2\beta\tan\frac{\hat{\varphi}_1(0)}{2}, \quad (42)$$

$$\chi = \alpha t + \mu_2 \tan \frac{\hat{\varphi}_1(0)}{2} \int \hat{\Phi}_1 dt, \tag{43}$$

and $c$ is a integration constant. Then,

$$\tan \frac{\hat{\varphi}_1}{2} = \tan \frac{\hat{\varphi}_1(0)}{2} + \frac{1}{\exp[-\chi]\{\int [\beta - \frac{1}{2}\mu_2 \hat{\Phi}_1] \exp[\chi] dt + c\}}. \tag{44}$$

From (42) one can see obviously how $\tan \frac{\hat{\varphi}_1}{2}$ is modulated by external magnetic flux through the super-conducting loops. Due to the mutual-inductance coupling, the two Josephson junctions should be considered in its totality. Then, for the 2nd Josephson junction, though it is not directly radiated by light, we can still obtain the time evolution of Cooper-pairs number operator $\hat{\mathcal{N}}_2$

$$\begin{aligned}\frac{d\hat{\mathcal{N}}_2}{dt} &= \frac{1}{i\hbar}\left[\hat{\mathcal{N}}_2, \mathcal{H}'\right] \\ &= \frac{1}{2e}\lambda_1 \hat{\mathcal{N}}_1 \frac{k^2}{M(1-k^2)},\end{aligned}$$

which shows the time evolution of Cooper-pairs number operator $\hat{\mathcal{N}}_2$ is affected by the inductive Cooper-pairs number operator $\hat{\mathcal{N}}_1$ due to the existence of the mutual-inductance coupling.

In summary, via the Hamilton dynamical approach we have constructed Hamiltonian for the mutual inductance coupling magnetic flux qubits. The entangled state representation is used to propose Cooper-pair number-phase quantization and the Hamiltonian operator for the whole system. The dynamical evolution of the phase difference operator and the Cooper-pairs number operator is investigated by Heisenberg equations of motion. We have shown how the magnetic flux affects the phase difference across the junction, which will be instructive for us to analyse the quantum manipulation.

## References


1. Pashkin, Yu.A., Yamamoto, T., Astafiev, O., Nakamura, Y., Averin, D.V., Tsai, J.S.: Nature **421**, 823 (2003)
2. Makhlin, Y., Schön, G., Shnirman, A.: Nature **398**, 305 (1999)
3. Vandersypen, L.M.K., Steffen, M., Breyta, G., Yannoni, C.S., Sherwood, M.H., Chuang, I.L.: Nature **414**, 883 (2001)
4. Gulde, S., Riebe, M., Lancaster, G.P.T., Becher, C., Eschner, J., Häfner, H., Schmidt-Kaler, F., Chuang, I.L., Blatt, R.: Nature **421**, 48 (2003)
5. Rodrigues, D.A., Spiller, T.P., Annett, J.F., Györffy, B.L.: J. Phys., Condens. Matter **19**, 436211 (2007)
6. Fan, H.Y., Wang, J.S., Liu, S.G.: Phys. Lett. A **359**, 580 (2006)
7. Fan, H.Y., Wang, J.S., Meng, X.G.: Int. J. Mod. Phys. B **21**, 3697 (2007)
8. Fan, H.Y., Wang, J.S., Fan, Y.: Mod. Phys. Lett. B **20**, 1041 (2006)
9. Liang, B.L., Wang, J.S., Fan, H.Y.: Chin. Phys. Lett. **24**, 3241 (2007)
10. Liang, B.L., Wang, J.S., Meng, X.G.: Int. J. Theor. Phys. **46**, 2901 (2007)
11. Makhlin, Y., Schön, G., Shnirman, A.: Rev. Mod. Phys. **73**, 357 (2001)
12. Vourdas, A.: Phys. Rev. B **49**, 12040 (1994)
13. Vourdas, A., Spiller, T.P.: Z. Phys. B **102**, 54 (1997)
14. Wang, J.S., Meng, X.G., Fan, H.Y.: Physica A **387**, 4453 (2008)
15. Fan, H.Y., Klauder, J.R.: Phys. Rev. A **49**, 704 (1994)
16. Fan, H.Y., Fan, Y.: Phys. Rev. A **54**, 95 (1996)
17. Fan, H.Y.: Int. J. Mod. Phys. B **17**, 2599 (2003)

18. Fan, H.Y.: Int. J. Mod. Phys. B **18**, 233 (2004)
19. Feynman, R.P., Leighton, R.B., Sands, M.: The Feynman Lectures on Physics, vol. 3. Addison-Wesley, Reading (1965)
20. Greiner, W., Reinhardt, J.: Field Quantization. Springer, Berlin (1996)
21. Einstein, A., Podolsky, B., Rosen, N.: Phys. Rev. **47**, 777 (1935)
22. Fan, H.Y.: Phys. Lett. A **289**, 172 (2001)
23. Fan, H.Y., Fan, Y., Song, T.Q.: Phys. Lett. A **305**, 222 (2002)